EXISTENCE OF STEADY SUBSONIC EULER FLOWS THROUGH INFINITELY LONG PERIODIC NOZZLES

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Abstract. In this paper, we study the global existence of steady subsonic Euler flows through infinitely long nozzles which are periodic in x_1 direction with the period L. It is shown that when the variation of Bernoulli function at some given section is small and mass flux is in a suitable regime, there exists a unique global subsonic flow in the nozzle. Furthermore, the flow is also periodic in x_1 direction with the period L. If, in particular, the Bernoulli function is a constant, we also get the existence of subsonic-sonic flows when the mass flux takes the critical value.

1. Introduction and Main Results

The study on subsonic and transonic flows in nozzles has grown enormously in recent years. Subsonic and subsonic-sonic potential flows in infinitely long nozzles were studied in [20, 21, 19, 15]. For full compressible Euler equations, Xie and Xin in [22] showed the existence of global subsonic flow in an infinitely long nozzle which tends to be flat at far field. The key point in [22] is to transform the system of Euler equations into a second order equation of stream function. The careful energy estimates give far field behavior and uniqueness of flows. The idea in [22] was generalized to subsonic Euler flows in axially symmetric nozzles [14]. Subsonic and subsonic-sonic potential flows past a body were studied in [3, 4, 7] and references therein. Subsonic Euler flows with nonzero vorticity in half space was investigated in [10]. Subsonic flows were also studied as a part of stability of transonic shock problem, see [8, 9, 5, 6, 11, 12, 17, 18, 23, 26] and references therein, where subsonic flows and nozzles are small perturbations of some given background flows and nozzles with simple geometries, respectively.

In this paper, we study the existence of global steady subsonic Euler flows through periodic nozzles. Consider 2-D steady isentropic Euler equations

$$(\rho u)_{x_1} + (\rho v)_{x_2} = 0, \tag{1}$$

$$(\rho u^2)_{x_1} + (\rho uv)_{x_2} + p_{x_1} = 0, \tag{2}$$

$$(\rho uv)_{x_1} + (\rho v^2)_{x_2} + p_{x_2} = 0, \tag{3}$$

where ρ, (u, v), and $p = p(\rho)$ denote the density, velocity, and pressure, respectively. In general, it is assumed that $p'(\rho) > 0$ for $\rho > 0$ and $p''(\rho) \geq 0$, where $c(\rho) = \sqrt{p'(\rho)}$ is called the sound speed. The most important examples include polytropic gases and isothermal gases. For polytropic gases, $p = A\rho^\gamma$ where A is a constant and γ is the adiabatic constant with $\gamma > 1$; and for isothermal gases, $p = c^2\rho$ with constant sound speed c [13].

We consider flows through an infinitely long periodic nozzle given by

$$\Omega = \{(x_1, x_2) | f_1(x_1) < x_2 < f_2(x_1), -\infty < x_1 < \infty\},$$

where f_i $(i = 1, 2)$ is $L-$periodic, i.e., $f_i(x_1 + L) = f_i(x_1)$ for $x_1 \in \mathbb{R}$. Suppose that there exist $\alpha \in (0, 1)$ and $C > 0$ such that

$$\|f_i\|_{C^{2,\alpha}(\mathbb{R})} \leq C \quad \text{and} \quad \inf_{x_1 \in [0,L]} (f_2(x_1) - f_1(x_1)) > 0. \tag{4}$$

Therefore, the domain Ω satisfies the uniform exterior sphere condition with some uniform radius $r > 0$. Without loss of generality, we assume that $f_1(0) = 0$ and $f_2(0) = 1$.

Suppose that the nozzle walls are impermeable so that the flow satisfies the no flow boundary condition

$$(u, v) \cdot \vec{\nu} = 0 \text{ on } \partial\Omega, \tag{5}$$

where $\vec{\nu}$ is the unit outward normal to the nozzle wall. It follows from (1) and (5) that

$$\int_l (\rho u, \rho v) \cdot \vec{n} dl \equiv m \tag{6}$$

holds for some constant m, which is called the mass flux, where l is any curve transversal to the x_1-direction, and $\vec{n}$ is the normal of l in the positive x_1-axis direction.

Using the continuity equation (1), when the flow is away from the vacuum, the momentum equations (2) and (3) are equivalent to

$$uu_{x_1} + vu_{x_2} + h(\rho)_{x_1} = 0, \tag{7}$$

$$uv_{x_1} + vv_{x_2} + h(\rho)_{x_2} = 0, \tag{8}$$

where $h(\rho)$ is the enthalpy of the flow satisfying $h'(\rho) = p'(\rho)/\rho$ and can be determined up to a constant. In this paper, for example, we always choose $h(0) = 0$ for polytropic gases and $h(1) = 0$ for isothermal gases. After determining this integral constant, we denote $H_0 = \inf_{\rho > 0} h(\rho)$.

It follows from (7) and (8) that

$$(u, v) \cdot \nabla(h(\rho) + \frac{1}{2}(u^2 + v^2)) = 0. \tag{9}$$

This implies that $\frac{u^2+v^2}{2}+h(\rho)$, which is called Bernoulli's function, is a constant along each streamline. For Euler flows in the nozzle, we assume that at $x_1 = 0$, Bernoulli function is given, i.e.,

$$\left(\frac{u^2+v^2}{2}+h(\rho)\right)(0,x_2) = B_0(x_2), \tag{10}$$

where $B_0(x_2)$ is a function defined on $[0,1]$.

When the Bernoulli function B is a constant, Proposition 3 shows that the flow is irrotational. The existence of periodic potential flows with small mass flux in periodic nozzles was obtained in [19]. In this paper, we first study the subsonic and subsonic-sonic periodic potential flows with relatively large mass flux.

Theorem 1. *If $B_0(x_2) \equiv \bar{B} > H_0$, then*

1. *there exists an $\hat{m} > 0$, such that for any $m \in (0,\hat{m})$ there exists a unique subsonic periodic flow $(\bar{\rho},\bar{u},\bar{v})$ which satisfies $\inf_{\bar{\Omega}} \bar{u} > 0$;*
2. *the maximum of Mach numbers of the flows increases as m increases and goes to one as $m \to \hat{m}$, i.e., the flows approach sonic;*
3. *there exist a sequence $m_n \to \hat{m}$ such that the associated potential flows $(\bar{\rho}_n,\bar{u}_n,\bar{v}_n)$ converge to $(\hat{\rho},\hat{u},\hat{v})$ almost everywhere, which satisfies*

$$\begin{cases} \nabla\times(\hat{u},\hat{v}) = 0, \\ div(\hat{\rho}\hat{u},\hat{\rho}\hat{v}) = 0, \end{cases} \tag{11}$$

   and the boundary condition (5) in the sense of divergence measure field, where $\hat{\rho}$ is determined by $\hat{u}$ and $\hat{v}$ via Bernoulli law.

When B_0 is not a constant, we have the following results on subsonic Euler flows in periodic nozzles.

Theorem 2. *Let the nozzle satisfy (4) and B_0 in (10) satisfy*

$$B_0'(0) \geq 0, \quad B_0'(1) \leq 0. \tag{12}$$

For any $m \in (0,\hat{m})$, there exists $\epsilon_0 > 0$ such that if $B_0(x_2)$ satisfies

$$\|B_0 - \bar{B}\|_{C^{1,1}([0,1])} = \epsilon \leq \epsilon_0, \tag{13}$$

where $\bar{B}$ is the constant in Theorem 1, then

1. *(Existence) there exists a periodic flow, i.e.,*

$$\rho(x_1+L,x_2)=\rho(x_1,x_2), u(x_1+L,x_2)=u(x_1,x_2), v(x_1+L,x_2)=v(x_1,x_2),$$

which satisfies the original Euler equations (1)-(3), the boundary condition (5), mass flux condition (6), and the condition (10);

2. *(Subsonic flows and positivity of horizontal velocity) the flow is globally uniformly subsonic and has positive horizontal velocity. More precisely, there are two positive constants* c_1 *and* c_2 *such that*

$$\sup_{\bar\Omega}(u^2+v^2-c^2(\rho)) < -c_1 \quad \textit{and} \quad u > c_2; \tag{14}$$

3. *(Regularity) the flow satisfies*

$$\|\rho\|_{C^{1,\alpha}(\Omega)}, \|u\|_{C^{1,\alpha}(\Omega)}, \|v\|_{C^{1,\alpha}(\Omega)} \le C$$

for some constant $C>0$*;*

4. *(Uniqueness) the flow is unique in the class of the periodic flows satisfying* (14).

A remark about Theorem 2 is as follows.

Remark 1. Using the analysis in this paper, it is easy to show that there exists a subsonic full compressible Euler flow in the nozzle, if the entropy is also prescribed at $x_1=0$.

The rest of the paper is organized as follows: In Section 2, we introduce the stream function formulation for the Euler equations and give the proof of Theorem 1. In Section 3, a boundary value problem for stream function is analyzed. This is divided into two steps. The existence of solutions for the associated problem is studied in Section 3.1. The uniqueness, periodicity, and positivity of horizontal velocity of the flows are proved in Section 3.2. In Section 4, we use the fixed point theorem to show the existence of Euler flows; the uniqueness of these flows are obtained by the energy method.

2. Stream Function Formulation of the Euler Flows and Proof of Theorem 1

2.1. **Bernoulli's Law and A New Formulation of the Euler Equations.** We recall that the steady Euler system for subsonic flows is a hyperbolic-elliptic coupled system [22]. Therefore, one has to resolve the hyperbolic mode.

To overcome the difficulties mentioned above, we introduce the stream function for 2-D steady compressible Euler flows so that the Bernoulli function can be reduced to a

single-valued function of steam function. This gives an equivalent formulation for Euler flows in terms of stream function.

Proposition 3. *For a smooth flow away from vacuum with no stagnation point, i.e.,* $u^2+v^2>0$*, the Euler system* (1)-(3) *is equivalent to the system of equations* (1), (9), *and*

$$\omega=\frac{v\partial_{x_1}B-u\partial_{x_2}B}{u^2+v^2},\tag{15}$$

where $B=\frac{1}{2}(u^2+v^2)+h(\rho)$ *and* $\omega=\partial_{x_1}v-\partial_{x_2}u$ *are Bernoulli function and vorticity, respectively.*

Proof: Let us first show that (1)-(3) implies (15). Differentiating the Bernoulli function with respect to x_1 and x_2, respectively, gives

$$\partial_1 B=u\partial_1 u+v\partial_1 v+\partial_1 h(\rho),\quad \partial_2 B=u\partial_2 u+v\partial_2 v+\partial_2 h(\rho).\tag{16}$$

This, together with (7)-(8), yields

$$\partial_1 B=v(\partial_1 v-\partial_2 u)=v\omega,\quad \partial_2 B=-u(\partial_1 v-\partial_2 u)=-u\omega.\tag{17}$$

Therefore, the equation (15) holds provided $u^2+v^2>0$.

Conversely, it follows from straightforward computation that (9) and (15) imply (17). Substituting (17) into (16) gives (7) and (8). Using (1), one has (2) and (3).

This finishes the proof of the proposition. $\square$

The continuity equation (1) implies that there exists a stream function ψ such that

$$\partial_{x_1}\psi=-\rho v,\ \ \partial_{x_2}\psi=\rho u.$$

Hence, for the flows away from the vacuum, (9) is equivalent to

$$\nabla^{\perp}\psi\cdot\nabla B=0,$$

where $\nabla^{\perp}=(-\partial_{x_2},\partial_{x_1})$. This yields that B and ψ are functionally dependent. Therefore, one may regard B as a function of ψ. We denote this function by $B=\mathcal{B}(\psi)$. It follows from the no flow boundary condition (5), that the nozzle walls are streamlines, so ψ is constant on each nozzle wall. Taking (6) into account, one may assume that

$$\psi=0 \text{ on } S_1,\ \text{and } \psi=m \text{ on } S_2,\tag{18}$$

where $S_i=\{(x_1,f_i(x_1))|x_1\in\mathbb{R}\}$ $(i=1,\,2)$.

As in [22], for any given $s > H_0$, there exist $\bar{\varrho} = \bar{\varrho}(s)$, $\varrho = \varrho(s)$ and $\Gamma = \Gamma(s)$ such that

$$h(\bar{\varrho}(s)) = s,\ h(\varrho(s)) + \frac{\Gamma^2(s)}{2} = s,\ \text{and}\ p'(\varrho(s)) = \Gamma^2(s),$$

where $\bar{\varrho}(s)$, $\varrho(s)$, and $\Gamma(s)$ are the maximum density, the critical density, and the critical speed, respectively for the states with given Bernoulli constant s. Set

$$\Sigma(s) = \varrho(s)\sqrt{2(s - h(\varrho(s)))}. \tag{19}$$

Then the straightforward calculations show that

$$\frac{d\bar{\varrho}}{ds} > 0, \quad \frac{d\varrho}{ds} > 0, \quad \text{and} \quad \frac{d\Sigma}{ds} > 0.$$

Obviously, $\varrho(s) < \bar{\varrho}(s)$, if $s > H_0 = \inf_{\rho>0} h(\rho)$. Furthermore, there exists a $\bar{\delta} > 0$ such that

$$\bar{\varrho}(s_1) \geq \varrho(s_2) \quad \text{for any} \quad s_1, s_2 \in (\bar{B} - \bar{\delta}, \bar{B} + \bar{\delta}). \tag{20}$$

For a fixed s, if ρ and $\mathcal{M}$ satisfies

$$h(\rho) + \frac{\mathcal{M}}{2\rho^2} = s, \tag{21}$$

then ρ is a two-valued function of $\mathcal{M}$ for $\mathcal{M} \in (0, \Sigma^2(s))$ and the subsonic branch satisfies $\rho > \varrho(s)$, see [22]. When s varies, the subsonic branch will be denoted by

$$\rho = H(\mathcal{M}, s) \text{ for } (\mathcal{M}, s) \in \{(\mathcal{M}, s) | \mathcal{M} \in (0, \Sigma^2(s)), s > H_0\}. \tag{22}$$

In view of (21), we have

$$\frac{\partial H}{\partial s} = \frac{H^3}{H^2c^2 - \mathcal{M}} > 0, \quad \frac{\partial H}{\partial \mathcal{M}} = \frac{H}{2(\mathcal{M} - H^2c^2)} < 0.$$

2.2. **Potential Flows and Proof of Theorem 1.** If $B_0(x) \equiv \bar{B}$, we have $\mathcal{B}(\psi) = \bar{B}$. It follows from (22) that $\rho = H(|\nabla\psi|^2, \bar{B})$. Furthermore, (15) implies $\omega \equiv 0$ because $B(x_1, x_2) \equiv \bar{B}$ in Ω. Therefore, ψ satisfies

$$\text{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2, \bar{B})}\right) = 0. \tag{23}$$

Let $\{M_n\}$ be a strictly increasing sequence satisfying $\lim_{n\to\infty} M_n = \Sigma^2(\bar{B})$. We define $H_n(\cdot, \bar{B}) \in C^\infty(\mathbb{R})$ satisfying

$$H_n(M, \bar{B}) = \begin{cases} H(M, \bar{B}) & \text{if} \quad M \leq M_n, \\ H(M_n, \bar{B}) & \text{if} \quad M \geq \dfrac{M_n + \Sigma^2(\bar{B})}{2}. \end{cases} \tag{24}$$

Combining Lemmas 2.1 and 3.1 in [20] and Lemma 1 in [19] (or Theorem 2 in [2]), there exists a unique periodic solution $\bar{\psi}(\cdot;t)$ of the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla\psi}{H_n(|\nabla\psi|^2,\bar{B})}\right)=0, \\ \psi=0 \quad \text{on} \quad S_1, \quad \psi=t \quad \text{on} \quad S_2 \end{cases}$$

for $t\geq 0$. Define $\mathscr{M}_n(t)=\sup_{\bar{\Omega}}|\nabla\bar{\psi}_n(\cdot;t)|^2$. Then that $\mathscr{M}_n(t)$ is a continuous function of t follows from the same argument in Lemma 4.1 in [20]. Let $m_n=\sup\{t|\mathscr{M}_n(t)<M_n\}$ and $\hat{m}=\sup_n m_n$. Then as $m\to\hat{m}$, the maximum of $|\nabla\bar{\psi}|$ of solutions of the problem (23) and (18) tend to $\Sigma(\bar{B})$, i.e., the flows approach the sonic state. Moreover, Using the compensated compactness framework in [7] (Theorem 2.1) and [20] (Theorem 5.1), there exist a sequence $\{m_n\}\uparrow\bar{m}$ such that the associated $\bar{u}_n=\frac{\partial_2\bar{\psi}_n}{H(|\nabla\bar{\psi}_n|^2,\bar{B})}$, $\bar{v}_n=-\frac{\partial_1\bar{\psi}_n}{H(|\nabla\bar{\psi}_n|^2,\bar{B})}$, and $\bar{\rho}_n=H(|\nabla\bar{\psi}|^2,\bar{B})$ satisfy $\bar{u}_n\to\hat{u}$, $\bar{v}_n\to\hat{v}$, and $\bar{\rho}_n\to\hat{\rho}$ a.e., where $(\hat{u},\hat{v})$ satisfies $\sup_{\bar{\Omega}}\frac{\hat{u}^2+\hat{v}^2}{p'(\hat{\rho})}=1$, the system (11), and the boundary condition (5) in the sense of divergence measure field.

The proof of Theorem 1 finishes after we prove the following lemma on the properties of subsonic potential flows.

Lemma 4. *If $m>0$, then the solution $\bar{\psi}$ of the problem* (23) *and* (18) *satisfies* $\inf_{\bar{\Omega}}\partial_2\bar{\psi}>0$. *Furthermore, as m increases,* $\max_{\bar{\Omega}}|\nabla\bar{\psi}|$ *also increases.*

Proof: Note that $\bar{\psi}$ satisfies $0\leq\bar{\psi}\leq m$, therefore, $\bar{\psi}$ achieves its minimum and maximum on the boundaries S_1 and S_2 respectively. It follows from the Hopf lemma (Lemma 3.4 in [16]) that $\partial_{x_2}\bar{\psi}>0$ on $\partial\Omega$. Therefore, the continuity of $\partial_{x_2}\bar{\psi}$ yields

$$\inf_{\substack{(x_1,x_2)\in\partial\Omega,\\ 0\leq x_1\leq L}}\partial_{x_2}\bar{\psi}>0.$$

If $\bar{\psi}$ satisfies (23), then $\partial_{x_2}\bar{\psi}$ satisfies

$$\partial_i(\bar{a}_{ij}\partial_j(\partial_{x_2}\bar{\psi}))=0, \quad \text{where} \quad \bar{a}_{ij}=\frac{(H^2c^2-|\nabla\bar{\psi}|^2)\delta_{ij}+\partial_i\bar{\psi}\partial_j\bar{\psi}}{H(|\nabla\bar{\psi}|^2-H^2c^2)}. \tag{25}$$

Here and later on, the repeated index means summation from 1 to 2. It follows from the strong maximum principle (Theorem 3.5 in [16]) that $\partial_2\bar{\psi}\geq 0$ on $\bar{\Omega}$. In fact, if the minimum of $\partial_{x_2}\bar{\psi}$ is achieved at some point $(x_1^*,x_2^*)\in\Omega$, by periodicity, we can always assume $|x_1^*|\leq L$. Then, it contradicts with the strong maximum principle for the equation (25) in the domain $\{(x_1,x_2)\in\Omega:|x_1|\leq\frac{3L}{2}\}$.

It follows from Bernstein estimates (Theorem 15.1 in [16]) and periodicity of the solutions that

$$\max_{\bar{\Omega}} |\nabla\bar{\psi}| \leq \max_{\partial\Omega} |\nabla\bar{\psi}|. \tag{26}$$

Thus using comparison principle, Hopf Lemma and periodicity, we can show that the maximum of the flow speed increases as m increases (cf. Lemma 4.4 in [20]). □

2.3. **Stream Function Formulation of General Euler Flows.** For any $m \in (0, \hat{m})$, there exists a unique periodic solution $\bar{\psi}(x_1, x_2) \in C^{2,\alpha}(\bar{\Omega})$ for the problem (23) and (18). Furthermore, there exist positive constants σ_0 and σ_1 such that

$$0 < \inf_{\bar{\Omega}} \bar{\psi}_{x_2} = \sigma_0 \leq \sigma_1 = \sup_{\bar{\Omega}} |\nabla\bar{\psi}|. \tag{27}$$

Given $W \in \mathcal{S}$ defined by

$$\mathcal{S} = \{W \in C^{1,\beta}([0,1]),\ \int_0^1 W(s)ds = m,\ \|W - \bar{\psi}_{x_2}(0,\cdot)\|_{C^{1,\beta}[0,1]} \leq \sigma_0/2\},$$

where $\beta \in (0, \alpha)$, then $W(s) > \sigma_0/2$ for $s \in [0,1]$. Therefore there exist a function $y = \kappa(\psi)$ such that

$$\psi = \int_0^{\kappa(\psi)} W(s)ds. \tag{28}$$

Differentiating (28) with respect to ψ yields

$$\kappa'(\psi) = \frac{1}{W(\kappa(\psi))}. \tag{29}$$

This shows that $\kappa(\psi) \in C^{2,\beta}([0,m])$. Suppose that the Bernoulli function at $x_1 = 0$ is $B_0(\kappa(\psi))$ in terms of ψ, since the Bernoulli function is a constant along each stream line, we get the Bernoulli function defined in the whole domain Ω by $B(x_1, x_2) = \mathcal{B}(\psi(x_1, x_2)) = B_0(\kappa(\psi(x_1, x_2)))$. Define $\mathcal{B}(\psi) = B_0(\kappa(\psi))$. Then $\mathcal{B}(\psi) \in C^2([0,m])$. Combining (12) and (29) gives

$$\|\mathcal{B} - \bar{B}\|_{C^{1,1}([0,m])} \leq C\epsilon, \quad \mathcal{B}'(0) \geq 0 \quad \text{and} \quad \mathcal{B}'(m) \leq 0. \tag{30}$$

Therefore, $\rho = H(|\nabla\psi|^2, \mathcal{B}(\psi))$ follows from (22) for subsonic flows. The equation (15) becomes the following second order equation for the stream function ψ,

$$\operatorname{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2, \mathcal{B}(\psi))}\right) = H(|\nabla\psi|^2, \mathcal{B}(\psi))\mathcal{B}'(\psi). \tag{31}$$

We first solve the equation (31) with the boundary condition (18). Second, we define a map from W to $\psi_{x_2}(0,\cdot)$. The fixed point of this map will induce the existence of the solutions of Euler equations.

3. Analysis of the Boundary Value Problem for Stream Function

3.1. **Existence of Solutions.** There are two main difficulties to solve the equation (31) in Ω. The first difficulty is that the equation (31) becomes degenerate elliptic at sonic states. In addition, H is not well-defined for arbitrary ψ and $|\nabla\psi|$; neither is $\mathcal{B}$. The second difficulty is that this is a problem in an unbounded domain. Our basic strategy is that we extend the definition of $\mathcal{B}$ appropriately, truncate $|\nabla\psi|$ appeared in H in a suitable way, and use a sequence of problems in bounded domains to approximate the problem (31) and (18).

Set

$$\tilde{g}(s)=\begin{cases}\mathcal{B}'(s), & \text{if } 0\le s\le m,\\ \mathcal{B}'(m)(2m-s)/m, & \text{if } m\le s\le 2m,\\ \mathcal{B}'(0)(s+m)/m, & \text{if } -m\le s\le 0,\\ 0, & \text{if } s\ge 2m \text{ or } s\le -m.\end{cases}$$

It is obvious that $\tilde{g}\in C^{0,1}(\mathbb{R})$ and

$$\|\tilde{g}(s)\|_{C^{0,1}(\mathbb{R}^1)}\le\|\mathcal{B}'(s)\|_{C^{0,1}([0,m])}\le 2\epsilon/\sigma_0.$$

Define

$$\tilde{\mathcal{B}}(s)=\mathcal{B}(0)+\int_0^s\tilde{g}(t)dt.$$

Then, $\|\tilde{\mathcal{B}}'\|_{C^{0,1}(\mathbb{R}^1)}=\|\tilde{g}\|_{C^{0,1}(\mathbb{R}^1)}\le\|\mathcal{B}'\|_{C^1([0,m])}\le C\epsilon$. Therefore,

$$|\tilde{\mathcal{B}}(\psi)-\bar{B}|\le C\epsilon.$$

Hence, there exists $\tilde{\epsilon}_0>0$ such that if $0<\epsilon<\tilde{\epsilon}_0$, then $\tilde{\mathcal{B}}(\psi)>H_0=\inf_s h(s)$. In view of (30), $\tilde{\mathcal{B}}$ also satisfies

$$\tilde{\mathcal{B}}'(s)\ge 0\quad\text{for } s\le 0\quad\text{and}\quad\tilde{\mathcal{B}}'(s)\le 0\text{ for } s\ge m. \tag{32}$$

Let $\check{B}=\min_{x\in[0,1]}B_0(x)$. Choose θ_0 to be a fixed positive constant satisfying

$$0<\theta_0\le\min\{\Sigma^2(\check{B})/2,\Sigma^2(\bar{B})-\sigma_1^2\}, \tag{33}$$

where σ_1 is defined in (27). Let $\zeta \in C^\infty(\mathbb{R})$ satisfy

$$\zeta(s) = \begin{cases} s, & \text{if } s < -\theta_0/4, \\ -\theta_0/8, & \text{if } s \geq -\theta_0/8, \end{cases}$$

and define $\tilde{\rho} = H(\zeta(|\nabla\psi|^2 - \Sigma^2(\tilde{\mathcal{B}}(\psi)) + \Sigma^2(\tilde{\mathcal{B}}(\psi), \tilde{\mathcal{B}}(\psi))$. Instead of the equation (31), we begin with investigating the equation,

$$\partial_1(\tfrac{\partial_1\psi}{\tilde{\rho}}) + \partial_2(\tfrac{\partial_2\psi}{\tilde{\rho}}) = \tilde{\rho}\tilde{\mathcal{B}}'(\psi). \tag{34}$$

The equation (34) can also be written in the following non-divergence form

$$A_{ij}(\nabla\psi, \psi)\partial_{ij}\psi = \mathcal{F}(\nabla\psi, \psi) \tag{35}$$

where

$$A_{ij}(\nabla\psi, \psi)) = \tilde{\rho}\delta_{ij} + \frac{\zeta'\tilde{\rho}}{(\tilde{\rho}^2\tilde{c}^2 - (\zeta + \Sigma^2))}\partial_i\psi\partial_j\psi,$$

and

$$\mathcal{F}(\nabla\psi, \psi) = \tilde{\mathcal{B}}'(\psi)\left(\tilde{\rho}^3\frac{\tilde{\rho}^2\tilde{c}^2 - (\zeta + \Sigma^2) + |\nabla\psi|^2}{\tilde{\rho}^2\tilde{c}^2 - (\zeta + \Sigma^2)} + \frac{(\zeta' - 1)\Sigma\Sigma'|\nabla\psi|^2}{\tilde{\rho}^2\tilde{c}^2 - (\zeta + \Sigma^2)}\right),$$

where the variables in ζ and Σ are $|\nabla\psi|^2 - \tilde{\mathcal{B}}(\psi)$ and $\tilde{\mathcal{B}}(\psi)$, respectively. The direct calculation shows that the eigenvalues Λ and λ of $[A_{ij}]_{2\times 2}$ satisfy $C^{-1} \leq |\Lambda/\lambda| \leq C$ for some constant C and thus the equation (35) is uniformly elliptic. However, $\mathcal{F}$ involves a quadratic growth in $|\nabla\psi|$, so it is not easy to apply the classical elliptic theory directly. The strategy is to modify $\mathcal{F}$ by

$$\tilde{\mathcal{F}}(\nabla\psi, \psi) = \tilde{\mathcal{B}}'(\psi)\frac{\tilde{\rho}^5\tilde{c}^2}{\tilde{\rho}^2\tilde{c}^2 - (\zeta + \Sigma^2)}, \tag{36}$$

It is easy to see that $\tilde{\mathcal{F}} = \mathcal{F}$ when ψ satisfies $|\nabla\psi|^2 - \Sigma^2(\tilde{\mathcal{B}}(\psi)) \leq -\theta_0/4$.

Proposition 5. *There exists a solution $\psi(\cdot; t) \in C^{2,\alpha}(\bar{\Omega})$ of the problem*

$$\begin{cases} A_{ij}\partial_{ij}\psi = \tilde{\mathcal{F}} \\ \psi = 0 \quad \text{on } S_1, \quad \psi = t \text{ on } S_2. \end{cases} \tag{37}$$

Furthermore, if $t \leq m$, then

$$0 \leq \psi \leq m. \tag{38}$$

There exists $m_1 > 0$ and $\epsilon_1 > 0$ such that if $0 \leq t \leq m_1$ and $\|\mathcal{B}'\| \leq \epsilon_1$, then

$$|\nabla\psi|^2 - \Sigma(\mathcal{B}(\psi)) \leq -\theta_0/2. \tag{39}$$

Proof: For the unbounded domain Ω, we use a sequence of boundary value problem defined in bounded domains $\tilde{\Omega}_N$ to approximate the problem in Ω, where $\tilde{\Omega}_N \in C^{2,\alpha}$ satisfies $\Omega_N \subset \tilde{\Omega}_N \subset \Omega_{2N}$ with $\Omega_k := \Omega \cap \{|x_1| \le k\}$. The construction of $\tilde{\Omega}_N$ can be found in [20].

We first solve the boundary value problem

$$\begin{cases} A_{ij}(\nabla\psi,\psi)\partial_{ij}\psi = \tilde{\mathcal{F}}(\nabla\psi,\psi) \text{ in } \tilde{\Omega}_N, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} t \text{ on } \partial\tilde{\Omega}_N. \end{cases} \tag{40}$$

Similar to the proof of Proposition 3 in [22], there exists a $C^{2,\alpha}$-solution ψ_N, such that

$$|\psi_N| \le C\left(t + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right), \quad \|\psi_N\|_{2,\alpha;\tilde{\Omega}_N} \le C\left(\Lambda/\lambda, |f_i|_{C^{2,\alpha}}, m, \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right),$$

By Arzela-Ascoli theorem, one can select a subsequence of $\{\psi_N\}$ (We still label it by $\{\psi_N\}$), such that

$$\psi_N \to \psi \text{ in } C^{2,\alpha_1}(K) \text{ with } K \Subset \bar{\Omega}, \ \alpha_1 < \alpha.$$

Obviously, ψ solves (37).

Since $\tilde{\mathcal{B}}$ satisfies (32), then $\widetilde{\mathcal{F}}(\nabla\psi_N,\psi_N) \ge 0$ in the domain $\tilde{\Omega}_N \cap \{\psi_N \ge m\}$. Thus according to system (40),

$$A_{ij}(\nabla\psi_N,\psi_N)\partial_{ij}\psi_N \ge 0 \text{ in } \tilde{\Omega}_N \cap \{\psi_N \ge m\}.$$

Since

$$\psi_N \le m \text{ on } \partial\left(\tilde{\Omega}_N \cap \{\psi_N \ge m\}\right),$$

by the maximum principle (Theorem 3.1 in [16]), one has

$$\psi_N \le \sup_{\partial\tilde{\Omega}_N} \psi_N \le m \text{ in } \tilde{\Omega}_N \cap \{\psi_N \ge m\}.$$

Similarly, it is also true that

$$\psi_N \ge \inf_{\partial\tilde{\Omega}_N} \psi_N \ge 0 \text{ in } \tilde{\Omega}_N \cap \{\psi_N \le 0\}.$$

Therefore, one has $0 \le \psi_N \le m$. So the limit ψ satisfies (38).

The Hölder estimate for the gradients of elliptic equations of two variables implies that

$$[\psi_N]_{1,\mu;\tilde{\Omega}_N} = \sup_{x,y\in\tilde{\Omega}_N} \frac{|\nabla\psi_N(x) - \nabla\psi_N(y)|}{|x-y|^\mu} \le C(\Lambda/\lambda, |f_i|_2)\left(1 + t + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right).$$

Then, ψ_N satisfies the estimate

$$|\nabla\psi_N|^2 \le \eta\left(1+t+\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right)^2 + C_\eta\left(t+\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right)^2,$$

where C_η is independent of N. Note that

$$|\tilde{\mathcal{F}}| \le C\epsilon, \tag{41}$$

there exist η_1, m_1 and ϵ_1 such that

$$\eta_1\left(1+m_1+C\epsilon_1\right)^2 \le \min\{\frac{\Sigma^2(\check{B})}{4}, \frac{\sigma_1^2}{2}\}, \quad C_{\eta_1}\left(m_1+C\epsilon_1\right)^2 \le \min\{\frac{\Sigma^2(\check{B})}{4}, \frac{\sigma_1^2}{2}\}. \tag{42}$$

If $t \le m_1$, $\epsilon \le \epsilon_1$, then

$$|\nabla\psi_N|^2 - \Sigma^2(\mathcal{B}(\psi_N)) \le -\frac{\Sigma^2(\check{B})}{2} \le -\theta_0.$$

Thus, the limit ψ satisfies (39) if ϵ_1 is suitably small.

This finishes the proof of the proposition. □

3.2. **Uniqueness, Periodicity and Positivity of Horizontal Velocity.** In this subsection, we show that the uniformly subsonic solution obtained in Proposition 5 for the problem (31) with the boundary condition

$$\psi = 0 \quad \text{on} \quad S_1, \quad \text{and} \quad \psi = t \quad \text{on} \quad S_2. \tag{43}$$

is unique.

Proposition 6. *There exists a positive constant $\epsilon_2 < \epsilon_1$ such that if $\|\mathcal{B}'\|_{C^{0,1}} = \epsilon \le \epsilon_2$, then for $0 \le t \le m$, the solution ψ of (31) and* (43) *which satisfies*

$$|\nabla\psi|^2 - \Sigma(\mathcal{B}(\psi)) \le -\theta_0/4, \quad 0 \le \psi \le m \tag{44}$$

must be unique.

Proof: Let $\psi_i, i = 1, 2$, both solve (31) and (43). Set $\Psi = \psi_1 - \psi_2$. Then Ψ satisfies the following elliptic boundary value problem,

$$\begin{cases} \partial_i(a_{ij}\partial_j\Psi) + \partial_i(b_i\Psi) = c_i\partial_i\Psi + d\Psi, \text{ in } \Omega, \\ \Psi = 0, \text{ on } \partial\Omega, \end{cases} \tag{45}$$

where

$$a_{ij} = \int_0^1 \frac{(\tilde{H}^2\tilde{c}^2 - |\nabla\tilde{\Psi}|^2)\delta_{ij} + \partial_i\tilde{\Psi}\partial_j\widetilde{\Psi}}{(|\nabla\widetilde{\Psi}|^2 - \tilde{H}^2\tilde{c}^2)\tilde{H}} d\theta, \quad b_i = -\int_0^1 \frac{\partial_i\widetilde{\Psi}\tilde{H}\mathcal{B}'(\widetilde{\Psi})}{\widetilde{H}^2\tilde{c}^2 - |\nabla\widetilde{\Psi}|^2} d\theta,$$

$$c_i = \int_0^1 \frac{\tilde{H}\partial_i\tilde{\Psi}}{|\nabla\widetilde{\Psi}|^2 - \tilde{H}^2\tilde{c}^2}\mathcal{B}'(\tilde{\Psi})d\theta, \quad d = \int_0^1 \frac{\tilde{H}^3}{\widetilde{H}^2\tilde{c}^2 - |\nabla\widetilde{\Psi}|^2}[\mathcal{B}'(\widetilde{\Psi})]^2 + \tilde{H}\mathcal{B}''(\widetilde{\Psi})d\theta,$$

with $\tilde{c} = \sqrt{p'(\tilde{H})}$, $\tilde{H} = H(|\nabla\widetilde{\Psi}|^2, \mathcal{B}(\widetilde{\Psi}))$, and $\widetilde{\Psi} = \theta\psi_1 + (1-\theta)\psi_2$. Since $\|\mathcal{B}'\|_{C^{0,1}} \leq C\epsilon$, we have

$$|b_i| + |c_i| + |d| \leq C\epsilon.$$

Choose a smooth cut-off function $\eta \in C_0^\infty(\mathbb{R})$ satisfying

$$\eta = \eta(s) = \begin{cases} 0, & |s| \geq (N+1)L, \\ 1, & |s| \leq NL. \end{cases} \tag{46}$$

Multiplying $\eta^2(x_1)\Psi$ on both sides of the equation (45) and integrating by parts yields that

$$\begin{aligned}
\iint_{\{|x_1|\leq NL\}\cap\Omega} |\nabla\Psi|^2 dx_1dx_2 &\leq C\iint_\Omega \eta^2(a_{ij}\partial_i\Psi\partial_j\Psi)dx_1dx_2 \\
&\leq C\iint_\Omega (|\eta\eta'\Psi|(|\Psi| + |\nabla\Psi|) + \epsilon\eta^2|\Psi|(|\Psi| + |\nabla\Psi|))dx_1dx_2 \\
&\leq C\iint_{\{NL\leq|x_1|\leq(N+1)L\}\cap\Omega} (|\Psi|^2 + |\nabla\Psi|^2)dx_1dx_2 \\
&\quad + C\epsilon\iint_{\{|x_1|\leq NL\}\cap\Omega} (|\Psi|^2 + |\nabla\Psi|^2)dx_1dx_2.
\end{aligned}$$

Since $\Psi = 0$ on $\partial\Omega$, by Poincare's inequality, we have

$$\iint_{\{kL\leq|x_1|\leq(k+1)L\}\cap\Omega} |\Psi|^2 dx_1dx_2 \leq C\iint_{\{kL\leq|x_1|\leq(k+1)L\}\cap\Omega} |\nabla\Psi|^2 dx_1dx_2$$

for any $k \in \mathbb{Z}$. When $\epsilon \leq \epsilon_2$ for some $\epsilon_2 > 0$, we have

$$\iint_{\{|x_1|\leq NL\}\cap\Omega} |\nabla\Psi|^2 dx_1dx_2 \leq C\iint_{\{NL\leq|x_1|\leq(N+1)L\}\cap\Omega} |\nabla\Psi|^2 dx_1dx_2., \tag{47}$$

Noting that $|\nabla\Psi| \in L^\infty(\Omega)$, it yields that $\nabla\Psi \in L^2(\Omega)$. Thus

$$\lim_{N\to\infty}\iint_{\{NL\leq|x_1|\leq(N+1)L\}\cap\Omega} |\nabla\Psi|^2 dx_1dx_2 = 0.$$

Therefore, the estimate (47) implies that $\iint_\Omega |\nabla\Psi|^2 \equiv 0$. It follows from $\Psi \equiv 0$ on $\partial\Omega$ that $\Psi \equiv 0$ in Ω. Therefore, the solution of (31) and (43) is unique. $\square$

One can check easily that if $\psi(x_1, x_2)$ solves the boundary value problem (31) and (18), so does $\psi(x_1 + L, x_2)$. Then the uniqueness implies the following corollary.

Corollary 7. *For any $\theta_0 > 0$, there exists a positive constant $\epsilon_2 < \epsilon_1$ such that if $\|\mathcal{B}'\|_{C^{0,1}} = \epsilon \le \epsilon_2$, then for $0 \le t \le m$, the solution ψ of (31) and* (43) *satisfying* (44) *must be periodic with respect to x_1 with period L, i.e.,*

$$\psi(x_1, x_2) = \psi(x_1 + L, x_2), \forall (x_1, x_2) \in \Omega.$$

Now we show that $\psi_{x_2}(0, \cdot) \in \mathcal{S}$.

Proposition 8. *For any $\theta_0 > 0$, there exists a positive constant $\epsilon_3 < \epsilon_2$ such that if $\|\mathcal{B}'\|_{C^{0,1}} = \epsilon \le \epsilon_3$, then for $0 \le t \le m$, the solution ψ of (31) and* (43) *satisfying* (44), *then $\|\nabla\psi - \nabla\bar{\psi}\|_{C^{1,\alpha}} \le C\epsilon$.*

Proof: Set $\Psi = \psi - \bar{\psi}$. Subtracting (23) from (31) gives that Ψ satisfies

$$\begin{cases} \partial_i(a_{ij}\partial_j\Psi + b_i(\mathcal{B}(\psi) - \bar{B})) = H(|\nabla\psi|^2, \mathcal{B}(\psi))\mathcal{B}'(\psi), \quad \text{in} \quad \Omega, \\ \Psi = 0 \quad \text{on} \quad \Omega, \end{cases} \tag{48}$$

where

$$a_{ij} = \int_0^1 \frac{(\hat{H}^2\hat{c}^2 - |\nabla\hat{\Psi}|^2)\delta_{ij} + \partial_i\hat{\Psi}\partial_j\hat{\Psi}}{\hat{H}(\hat{H}^2\hat{c}^2 - |\nabla\hat{\Psi}|^2)} d\theta, \quad b_i = -\int_0^1 \frac{\partial_i\hat{\Psi}\hat{H}}{\hat{H}^2\hat{c}^2 - |\nabla\hat{\Psi}|^2} d\theta,$$

with $\hat{c} = \sqrt{p'(\hat{H})}$, $\hat{H} = H(|\nabla\hat{\Psi}|^2, \theta\mathcal{B}(\psi) + (1-\theta)\bar{B})$, $\hat{\Psi} = \theta\psi + (1-\theta)\bar{\psi}$. Since both ψ and $\bar{\psi}$ are periodic, Ψ is also periodic with period L. Multiplying the equation in (48) with Ψ and integrating the resulting equation on $\Omega \cap \{-2L \le x_1 \le 2L\}$, and integration by parts yield

$$\|\nabla\Psi\|_{L^2(\Omega\cap\{-2L\le x_1\le 2L\})} \le C\epsilon.$$

Applying Moser's iteration (Theorems 8.17 and 8.25 in [16]), we have

$$\|\Psi\|_{L^\infty(\Omega\cap\{-\frac{L}{2}\le x_1\le\frac{3L}{2}\})} \le C\epsilon.$$

Using the estimate for elliptic equation of two variables (Theorem 12.4 and global estimates on page 304 in [16]), we have

$$\|\Psi\|_{C^{1,\alpha}(\bar{\Omega}\cap\{0\le x_1\le L\})} \le C\epsilon.$$

Using Schauder estimate, we have $\|\Psi\|_{C^{2,\alpha}(\Omega_1)} \le C\epsilon$. Choosing $\epsilon_3 > 0$ sufficiently small shows that $\|\nabla\Psi\|_{C^{1,\alpha}(\Omega_1)} \le \sigma_0/2$ provided $0 < \epsilon \le \epsilon_3$. This finishes the proof of the

proposition. □

4. Existence and Uniqueness of the Euler Flows

In this section, We prove the existence and uniqueness of subsonic solutions for Euler equations.

Proposition 9. *There exists a positive constant ϵ_5 such that if $\|B_0'\|_{C^1} \leq \epsilon_5$, then the system* (1)-(3) *under the conditions (5), mass flux condition (6), and the condition (10) has a subsonic solution with positive horizontal velocity.*

Proof: It follows from Proposition 5 that there exists a solution $\psi(\cdot;t)$ for the problem (37). Set

$$S(t) = \{\psi(\cdot;t)|\psi(\cdot;t) \text{ solves } (37)\}$$

and

$$\Delta(t) = \inf_{\psi\in S(t)} \sup_{\bar{\Omega}} \{|\nabla\psi(\cdot;t)|^2 - \Sigma^2(\tilde{B}(\psi(\cdot;t)))\}$$

Let $\tilde{m} = \sup_s\{s \in (0,\hat{m})|\Delta(s) \leq -\theta_0/4\}$. Then $\Delta(t)$ is continuous for $t \in [0,\tilde{m}]$(cf. Proposition 6 in [22]). We claim that $\tilde{m} > m$. Indeed, it follows from Proposition 5 that $\Delta(m_1) \leq -\theta_0/2$. If $\tilde{m} \leq m$, then

$$\begin{aligned}
&|\nabla\psi(\cdot;\tilde{m})|^2 - \Sigma^2(\mathcal{B}(\psi(\cdot;\tilde{m})))\\
=&|\nabla\psi(\cdot;\tilde{m})|^2 - |\nabla\bar{\psi}(\cdot;\tilde{m})|^2 + |\nabla\bar{\psi}(\cdot;\tilde{m})|^2 - \Sigma^2(\bar{B}) + \Sigma^2(\bar{B}) - \Sigma^2(\mathcal{B}(\psi(\cdot;\tilde{m})))\\
\leq&C\epsilon - \theta_0 + C\epsilon,
\end{aligned}$$

where we use the property that the maximum of flow speed increases as the mass flux increases in Lemma 4. Thus there exists a positive $\epsilon_4 \leq \epsilon_3$ such that $|\nabla\psi(\cdot;\tilde{m})|^2 - \Sigma^2(\mathcal{B}(\psi(\cdot;\tilde{m}))) \leq -\theta_0/2$. This contradicts with $\Delta(\tilde{m}) = -\theta_0/4$ which follows from the continuity of $\Delta(s)$. Thus $\tilde{m} > m$. Thus it implies that for $t = m$ the problem (37) has a unique solution satisfying

$$|\nabla\psi|^2 - \Sigma(\mathcal{B}(\psi)) \leq -\theta_0/4.$$

Using Proposition 8, there exists a positive constant $\epsilon_4 < \epsilon_3$ such that $|\nabla\psi - \nabla\bar{\psi}| \leq \sigma_0/2$. In particular, $|\partial_{x_2}\psi(0,\cdot) - \partial_{x_2}\bar{\psi}(0,\cdot)| \leq \sigma_0/2$. Therefore $\partial_{x_2}\psi(0,\cdot) \in \mathcal{S}$. Hence, we can define a map $T : \mathcal{S} \to \mathcal{S}$ by

$$T(W) = \psi_{x_2}(0,\cdot).$$

By Proposition 8, $\|\psi_{x_2}(0,\cdot)-\bar{\psi}_{x_2}(0,\cdot)\|_{C^{1,\alpha}}\le C\epsilon$. Thus $T\mathcal{S}$ is a compact subset of $\mathcal{S}$. It is easy to see that T is a continuous map. Note that $\mathcal{S}$ is a closed convex set in $C^{1,\beta}[0,1]$; the existence of fixed point of T on $\mathcal{S}$ follows from Schauder fixed point theorem (Theorem 11.1 in [16]).

Let $\hat{W}$ be the fixed point of T and $\hat{\kappa}(y)$ satisfies

$$y=\int_0^{\hat{\kappa}(y)}\hat{W}(s)ds.$$

Define $\hat{\mathcal{B}}(\psi)=B_0(\hat{\kappa}(\psi))$. Then

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla\psi}{H(|\nabla\psi|^2,\hat{\mathcal{B}}(\psi))}\right)=H(|\nabla\psi|^2,\hat{\mathcal{B}}(\psi))\hat{\mathcal{B}}'(\psi) \\ \psi=0 \text{ on } S_1 \text{ and } \psi=m \text{ on } S_2 \end{cases}$$

has a solution satisfies $\hat{\psi}_{x_2}(0,x_2)=\hat{W}(x_2)$. It is clear that $\rho=H(|\nabla\hat{\psi}|^2,\hat{\mathcal{B}}(\psi))$, $u=\partial_{x_2}\hat{\psi}/\rho$, and $v=-\partial_{x_2}\hat{\psi}/\rho$ satisfy the Euler equations and the condition

$$\left(\frac{u^2+v^2}{2}+h(\rho)\right)(0,x_2)=\hat{\mathcal{B}}(\psi(0,x_2))=B_0(x_2).$$

Thus we get the existence of solution for Euler system under the conditions (5), mass flux condition (6), and the condition (10). Furthermore, it follows from Proposition 8 that horizontal velocity is positive. □

Now we show that the periodic Euler flow is also unique.

Proposition 10. *There exists a positive constant ϵ_6 such that if $\|B_0'\|_{C^1}\le\epsilon_6$, then the uniformly subsonic solution the system* (1)-(3) *under the conditions (5), mass flux condition (6), and the condition (10) is unique.*

Proof: Suppose that ψ_i $(i=1,2)$ are stream functions of two periodic solutions of the Euler equations with positive velocity. Let κ_i $(i=1,2)$ be the functions satisfying $\kappa_i(\psi_i(0,x_2))=x_2$ $(i=1,2)$. Set $\mathcal{B}_i(\psi)=B_0(\kappa_i(\psi))$. Then ψ_i satisfies the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla\psi}{H(|\nabla\psi|^2,\mathcal{B}_i(\psi))}\right)=H(|\nabla\psi|^2,\mathcal{B}_i(\psi))\mathcal{B}_i'(\psi), \quad \text{in} \quad \Omega, \\ \psi=0 \quad \text{on} \quad S_1, \quad \psi=m \quad \text{on} \quad S_2. \end{cases}$$

Set $\Psi = \psi_1 - \psi_2$. Then Ψ satisfies the problem

$$\begin{cases} \partial_i(a_{ij}\partial_j\Psi) + \partial_i(b_i\mathcal{D}) = c_i\partial_i\Psi + d\mathcal{D} + e\mathcal{E}, \quad \text{in} \quad \Omega, \\ \Psi = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{49}$$

where

$$\begin{aligned} a_{ij} &= \int_0^1 \frac{(H^2c^2 - |\nabla\tilde{\Psi}|^2)\delta_{ij} + \partial_i\tilde{\Psi}\partial_j\tilde{\Psi}}{H(H^2c^2 - |\nabla\tilde{\Psi}|^2)}d\theta, \quad b_i = -\int_0^1 \frac{\partial_i\tilde{\Psi}H}{H^2c^2 - |\nabla\tilde{\Psi}|^2}d\theta \\ c_i &= \int_0^1 \frac{H\partial_i\tilde{\Psi}}{|\nabla\tilde{\Psi}|^2 - H^2c^2}(\theta\mathcal{B}_1'(\psi_1) + (1-\theta)\mathcal{B}_2'(\psi_2)) \\ d &= \int_0^1 \frac{H^3}{H^2c^2 - |\nabla\tilde{\Psi}|^2}(\theta\mathcal{B}_1'(\psi_1) + (1-\theta)\mathcal{B}_2'(\psi_2)d\theta, \quad e = \int_0^1 Hd\theta, \end{aligned}$$

with $\tilde{\Psi} = \theta\psi_1 + (1-\theta)\psi_2, H = H(|\nabla\tilde{\Psi}|^2, \theta\mathcal{B}_1(\psi_1) + (1-\theta)\mathcal{B}_2(\psi_2)$, $\mathcal{D} = \mathcal{B}_1(\psi_1) - \mathcal{B}_2(\psi_2)$ and $\mathcal{E} = \mathcal{B}_1'(\psi_1) - \mathcal{B}_2'(\psi_2)$.

We first multiply Ψ on both sides of the equation in (49) and integrate the resulting equation on Ω_1. In view of periodicity of the coefficients and Ψ in (49), integration by parts yields

$$\int_{\Omega_1} a_{ij}\partial_i\Psi\partial_j\Psi dx = \int_{\Omega_1} (b_i\mathcal{D}\partial_i\Psi + c_i\Psi\partial_i\Psi) + d\mathcal{D}\Psi + e\mathcal{E}\Psi dx. \tag{50}$$

Thus,

$$\textstyle\int_{\Omega_1} |\nabla\Psi|^2 dx \quad \leq \quad C\epsilon \int_{\Omega_1}(|\Psi|^2 + |\nabla\Psi|^2) + C\int_{\Omega_1}(|\mathcal{D}|^2 + |\mathcal{E}|^2)dx. \tag{51}$$

Since $\Psi \in W^{1,\infty}$, the first term on the right hand side of (51) is uniformly bounded.

Note that $\Psi = 0$ on $\partial\Omega \cap \bar{\Omega}_1$, Poincare inequality implies that

$$\int_{\Omega_1} |\Psi|^2 dx \leq C\int_{\Omega_1} |\nabla\Psi|^2 dx.$$

Therefore, the first term on the right hand side of (51) can be absorbed by the left hand side.

Now let us estimate the second term on the right hand of (51). First,

$$\begin{aligned} \int_{\Omega_1} |\mathcal{D}|^2 dx &= \int_{\Omega_1} |B_0 \circ \kappa_1(\psi_1) - B_0 \circ \kappa_2(\psi_2)|^2 dx \\ &\leq \int_{\Omega_1} |B_0 \circ \kappa_1(\psi_1) - B_0 \circ \kappa_1(\psi_2)|^2 dx + \int_{\Omega_1} |B_0 \circ \kappa_1(\psi_2) - B_0 \circ \kappa_2(\psi_2)|^2 dx \\ &= I_1 + I_2. \end{aligned}$$

Using mean value theorem, we have

$$I_1 \leq C\epsilon \int_{\Omega_1} |\Psi|^2 dx \leq C\epsilon \int_{\Omega_1} |\nabla\Psi|^2 dx.$$

Note that $\psi_1(0, \kappa_1(\psi_2)) = \psi_2(0, \kappa_2(\psi_2))$, we have

$$\int_0^{\kappa_1(\psi_2)} \partial_{x_2}\psi_1(0,s) - \partial_{x_2}\psi_2(0,s) ds = \int_{\kappa_1(\psi_2)}^{\kappa_2(\psi_2)} \partial_{x_2}\psi_2(0,s) ds.$$

In view of the fact that $\partial_{x_2}\psi_2 \geq \frac{\sigma_0}{2}$, we have

$$|\kappa_1(\psi_2) - \kappa_2(\psi_2)| \leq C\|\nabla\Psi(0,\cdot)\|_{L^2[0,1]}$$

Thus,

$$I_2 \quad \leq \quad \int_{\Omega_1} C\epsilon\|\nabla\Psi(0,\cdot)\|_{L^2[0,1]} dx_1 dx_2 \leq C\epsilon\|\nabla\Psi\|_{L^\infty(\Omega_1)}.$$

Similarly, we can show that $\int_{\Omega_1} |\mathcal{E}|^2 dx_1 dx_2 \leq C\epsilon\|\nabla\Psi\|_{L^\infty(\Omega_1)}$.

This implies that

$$\|\nabla\Psi\|_{L^2(\Omega_1)} \leq C\epsilon\|\nabla\Psi\|_{L^\infty(\Omega_1)}.$$

By Nash-Moser's iteration, we have

$$\|\Psi\|_{L^\infty(\Omega_1)} \leq C\epsilon\|\nabla\Psi\|_{L^\infty(\Omega_1)}$$

Using the estimate for elliptic equation of two variables (Theorem 12.4 and global estimates on page 304 in [16]), we have

$$\|\Psi\|_{C^{1,\alpha}(\Omega_1)} \leq C\epsilon\|\Psi\|_{L^\infty(\Omega_1)} = C\epsilon\|\Psi\|_{L^\infty(\Omega_1)}$$

Therefore $\Psi \equiv 0$ in Ω.

This finishes the proof of the proposition. □

Choosing $\epsilon_0 = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_5, \epsilon_6\}$, then Theorem 2 follows from Propositions 9 and 10.

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